

Braneworld Remarks in Riemann-Cartan Manifolds

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We analyze the projected effective Einstein equation in a 4-dimensional arbitrary manifold embedded in a 5-dimensional Riemann-Cartan manifold. The Israel-Darmois matching conditions are investigated, in the context where the torsion discontinuity is orthogonal to the brane. Unexpectedly, the presence of torsion terms in the connection does not modify such conditions whatsoever, despite of the modification in the extrinsic curvature and in the connection. Then, by imposing the \mathbb{Z}_2 -symmetry, the Einstein equation obtained via Gauss-Codazzi formalism is extended, in order to now encompass the torsion terms. We also show that the factors involving contorsion change drastically the effective Einstein equation on the brane, as well as the effective cosmological constant.

PACS numbers: 04.50.-h; 11.25.-w

I. INTRODUCTION

In the last years there has been an increasing interest in large extra dimension models [1], mainly due to the developments in string theory [2], but also to the possibility of the hierarchy problem explanation, presented for instance in Randall-Sundrum and Hořava-Witten braneworld scenarios [2, 3]. In particular, the Randall-Sundrum braneworld model [3] is effectively implemented in a 5-dimensional manifold (where there is one warped extra dimension) and it is based on a 5-dimensional reduction of Hořava-Witten theory [4, 5]. In Randall-Sundrum models, our universe is described by an infinitely thin membrane — the brane. One attempt of explaining why gravity is so weak is by trapping the braneworld in some higher-dimensional spacetime — the bulk — wherein the brane is considered as a submanifold. For instance, the observable universe proposed by Randall and Sundrum, in one of their two models, can be described as being a brane embedded in an AdS_5 bulk. There are several analogous models, which consider our universe as a D -dimensional braneworld embedded in a bulk of codimension one. In some models, there are some modifications in the scenario that allow the presence of a compact dimension on the brane [6]. It gives rise to the so called hybrid compactification.

As a crucial formal pre-requisite to try to describe gravity in a braneworld context, the bulk is imposed to present codimension one — in relation to the brane. There is a great amount of results applying the Gauss-Codazzi (GC) formalism [7] in order to derive the properties of such braneworld (see [8, 9] and references therein). In the case where the bulk has two more dimensions than

the brane, the GC formalism is no longer useful, since the concept of a thin membrane is meaningless, in the sense that it is not possible to define junction conditions in codimension greater than one. In such case the addition of a Gauss-Bonnet term seems to break the braneworld apparent sterility [10]. For higher codimensions, the situation is even worse.

Going back to the case of one non-compact extra dimension, after expressing the Einstein tensor in terms of the stress tensor of the bulk and extrinsic curvature corrections, it is necessary to develop some mechanism to explore some physical quantities on the brane. In order that the GC formalism to be useful, we must be able to express the quantities in the limit of the extra dimension going to zero — at the point where the brane is located. Using this procedure, two junction requirements [11], which are the well known Israel-Darmois matching conditions, emerge.

A formalism where a manifold, endowed with a connection presenting non-null torsion, is often required to describe physical theories that are more general in many aspects. For instance, torsion is essential when the description of fermionic matter coupled to gravity is considered, and corrections of higher order in the Einstein-Hilbert Lagrangian imply the presence of torsion in the theory. Also, in contrast to the Yang-Mills formalism, in the Poincaré gauge theory it is possible to construct an invariant action which is linear in field derivatives. This gives rise to the Einstein-Cartan theory as an immediate generalization of General Relativity (GR) in a Riemann-Cartan manifold [12]. In the absence of matter fields, Einstein-Cartan theory is equivalent to GR, and the invariant action in this case is exactly the Hilbert-Palatini action. Moreover, torsion also emerges in the interface between GR and gravity via string theory at low energy. In this vein, it seems quite natural to explore some aspects of braneworld models in the presence of torsion, which can be thought of as appearing in the

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theory as part of the connection gauge field and work in the Palatini formalism in the representation of orthonormal bases. In this case, the continuity of the projection of the connection field along the brane can be shown to follow from the consistency of the variational problem in the presence of the brane. For instance, as regards the variational problem in the presence of the brane one can look into the paper [13].

This paper is organized as follows: after presenting some geometric preliminaries involving Riemann-Cartan spacetimes in Section II, we introduce the concept of torsion in the context of GR and the Israel-Darmois matching conditions are investigated in the presence of torsion, in an approach that is similar to the formalism exhibited in reference [14]. In Section III, junction conditions are investigated in the context where the torsion discontinuity is orthogonal to the brane. Then, in Section IV the Gauss-Codazzi formalism is used in order to establish the role and implications of torsion terms in the braneworld framework scenario.

II. BRANEWORLD PRELIMINARIES

In this Section, we shall proceed as in ref. [14] presenting the fundamental setup necessary to develop the formalism concerning the matching conditions with torsion in the next Section, as well as the application of GC formalism in the last Section.¹

Hereon Σ denotes a D -dimensional Riemann-Cartan manifold modelling a brane embedded in a bulk, denoted by M . A vector space endowed with a constant signature metric, isomorphic to \mathbb{R}^{D+1} , can be identified at a point $x \in M$ as being the space $T_x M$ tangent to M , where M is locally diffeomorphic to its local foliation $\mathbb{R} \times \Sigma$. There always exists a 1-form field n , normal to Σ , which can be locally interpreted — in the case where n is timelike — as being cotangent to the worldline of observer families, i.e., the dual reference frame relative velocity associated with such observers.

Denote $\{e_a\}$ ($a = 0, 1, \dots, D$) a basis for the tangent space $T_x \Sigma$ at a point x in Σ , and naturally the cotangent space at $x \in \Sigma$ has an orthonormal basis $\{e^a\}$ such that $e^a(e_b) = \delta_b^a$. A reference frame at an arbitrary point in the bulk is denoted by $\{e_\alpha\}$ ($\alpha = 0, 1, 2, \dots, D+1$). When a local coordinate chart is chosen, it is possible to represent $e_\alpha = \partial/\partial x^\alpha \equiv \partial_\alpha$ and $e^\alpha = dx^\alpha$. The 1-form field orthogonal to the sections of $T\Sigma$ — the tangent bundle of Σ — can now be written as $n = n_\alpha e^\alpha$, and consider the Gaussian coordinate ℓ orthogonal to the section of $T\Sigma$, indicating how much an observer move out the D -dimensional brane into the $(D+1)$ -dimensional bulk. A vector field $v = v^\alpha e_\alpha$ in the bulk is split in components

in the brane and orthogonal to the brane, respectively as $v = v^a e_a + \ell e_{D+1}$. Since the bulk is endowed with a non-degenerate bilinear symmetric form g that can be written in a coordinate basis as $g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta$, the components of the metric in the brane and on the bulk are denoted respectively by $q_{\alpha\beta}$ and $g_{\alpha\beta}$, and related by

$$g_{\alpha\beta} = q_{\alpha\beta} + n_\alpha n_\beta. \quad (1)$$

The displacement away from the hypersurface, along one fixed but arbitrary geodesic, is given by $dx^\alpha = n^\alpha d\ell$, and in particular the expression $n_\alpha dx^\alpha = d\ell$ implies that $n^\alpha n_\alpha = \pm 1$, where $+1$ corresponds to a spacelike braneworld Σ , and -1 corresponds to a timelike Σ . The 1-form field n orthogonal to Σ , in the direction of increasing ℓ is given by $n = (\partial_\alpha \ell) e^\alpha$, and its covariant components are explicitly given by $n_\alpha = \partial_\alpha \ell$. Without loss of generality a timelike hypersurface Σ is taken, where a congruence of geodesics goes across it. Denoting the proper distance (or proper time) along these geodesics by ℓ , it is always possible to put $\ell = 0$ on Σ .

Denoting $\{x^\alpha\}$ a chart on both sides of the brane, define another chart $\{y^a\}$ on the brane. Here the same notation used in [14] is adopted, where Latin indices are used for hypersurface coordinates and Greek indices for coordinates in the embedding spacetime. The brane can be parametrized by $x^\alpha = x^\alpha(y^a)$, where the parametric index a runs over the dimensions of the hypersurface — not being a spacetime index — and the vierbein $h_a^\alpha := \frac{\partial x^\alpha}{\partial y^a}$ satisfy $h_a^\alpha n_\alpha = 0$. For displacements on the brane, it follows that

$$\begin{aligned} g &= g_{\alpha\beta} dx^\alpha \otimes dx^\beta = g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a} dy^a \right) \otimes \left(\frac{\partial x^\beta}{\partial y^b} dy^b \right) \\ &= q_{ab} dy^a \otimes dy^b, \end{aligned} \quad (2)$$

and so the induced metric components q_{ab} on Σ is related to $g_{\alpha\beta}$ by $q_{ab} = g_{\alpha\beta} h_a^\alpha h_b^\beta$.

Denoting by $[A] = \lim_{\ell \rightarrow 0^+} (A) - \lim_{\ell \rightarrow 0^-} (A)$ the change in a differential form field A across the braneworld Σ (wherein $\ell = 0$), the continuity of the chart x^α and ℓ across Σ implies that n_α and h_a^α are continuous, or, equivalently, $[n_\alpha] = [h_a^\alpha] = 0$.

Now, using the Heaviside distribution $\Theta(\ell)$ properties²

$$\Theta^2(\ell) = \Theta(\ell), \quad \Theta(\ell)\Theta(-\ell) = 0, \quad \frac{d}{d\ell} \Theta(\ell) = \delta(\ell),$$

the metric components $g_{\alpha\beta}$ can be written as distribution-valued tensor components

$$g_{\alpha\beta} = \Theta(\ell) g_{\alpha\beta}^+ + \Theta(-\ell) g_{\alpha\beta}^-,$$

where $g_{\alpha\beta}^+$ ($g_{\alpha\beta}^-$) denotes the metric on the $\ell > 0$ ($\ell < 0$) side of Σ . Differentiating the above expression, it reads

$$\partial_\gamma g_{\alpha\beta} = \Theta(\ell) \partial_\gamma g_{\alpha\beta}^+ + \Theta(-\ell) \partial_\gamma g_{\alpha\beta}^- + \delta(\ell) [g_{\alpha\beta}] n_\gamma.$$

¹ For a complete exposition concerning arbitrary manifolds and fiber bundles, see, e.g., [15, 16, 17, 18, 19].

² $\delta(\ell)$ is the Dirac distribution.

The last term is singular; moreover, this term creates problems when we compute the Christoffel symbols by generating the product $\Theta(\ell)\delta(\ell)$, which is not defined as a distribution. It can be shown that the condition $[g_{\alpha\beta}] = 0$ must be imposed for the connection to be defined as a distribution³, also implying the ‘first’ junction condition $[h_{ab}]$.

Besides a curvature associated with the connection that endows the bulk, in a Riemann-Cartan manifold the torsion associated with the connection is in general non zero. Its components can be written in terms of the connection components $\Gamma^\rho_{\beta\alpha}$ as

$$T^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha} - \Gamma^\rho_{\alpha\beta}. \quad (3)$$

The general connection components are related to the Levi-Civita connection components $\overset{\circ}{\Gamma}^\rho_{\alpha\beta}$ — associated with the spacetime metric $g_{\alpha\beta}$ components — through $\Gamma^\rho_{\alpha\beta} = \overset{\circ}{\Gamma}^\rho_{\alpha\beta} + K^\rho_{\alpha\beta}$, where $K^\rho_{\alpha\beta} = \frac{1}{2}(T^\rho_{\alpha\beta} + T^\rho_{\beta\alpha} - T^\rho_{\alpha\beta})$ denotes the contortion tensor components. It must be emphasized that curvature and torsion are properties of a connection, not of spacetime. For instance, the Christoffel and the general connections present different curvature and torsion, although they endow the very same manifold.

Now the distribution-valued Riemann tensor is calculated, in order to find the ‘second’ junction condition — the Israel matching condition. From the Christoffel symbols, it reads $\Gamma^\alpha_{\beta\gamma} = \Theta(\ell)\Gamma^{+\alpha}_{\beta\gamma} + \Theta(-\ell)\Gamma^{-\alpha}_{\beta\gamma}$, where $\Gamma^{\pm\alpha}_{\beta\gamma}$ are the Christoffel symbols obtained from $g^{\pm\alpha}_{\beta\gamma}$. Thus

$$\partial_\delta \Gamma^\alpha_{\beta\gamma} = \Theta(\ell)\partial_\delta \Gamma^{+\alpha}_{\beta\gamma} + \Theta(-\ell)\partial_\delta \Gamma^{-\alpha}_{\beta\gamma} + \delta(\ell)[\Gamma^\alpha_{\beta\gamma}]n_\delta,$$

and the Riemann tensor is given by $R^\alpha_{\beta\gamma\delta} = \Theta(\ell)R^{+\alpha}_{\beta\gamma\delta} + \Theta(-\ell)R^{-\alpha}_{\beta\gamma\delta} + \delta(\ell)A^\alpha_{\beta\gamma\delta}$, where $A^\alpha_{\beta\gamma\delta} = [\Gamma^\alpha_{\beta\delta}]n_\gamma - [\Gamma^\alpha_{\beta\gamma}]n_\delta$.

The next step is to find an explicit expression for the tensor $A^\alpha_{\beta\gamma\delta}$.

III. JUNCTION CONDITIONS WITH THE TORSION DISCONTINUITY ORTHOGONAL TO THE BRANE

Observe that the continuity of the metric across Σ implies that the tangential derivatives of the metric must be also continuous. If $\partial_\gamma g_{\alpha\beta} \equiv g_{\alpha\beta,\gamma}$ is indeed discontinuous, this discontinuity must be directed along the normal vector n^α . It is therefore possible to write

$$[g_{\alpha\beta,\gamma}] = \kappa_{\alpha\beta}n_\gamma,$$

for some tensor $\kappa_{\alpha\beta}$ (given explicitly by $\kappa_{\alpha\beta} = [g_{\alpha\beta,\gamma}]n^\gamma$). Then it follows that

$$[\overset{\circ}{\Gamma}^\alpha_{\beta\gamma}] = \frac{1}{2}(\kappa^\alpha_\beta n_\gamma + \kappa^\alpha_\gamma n_\beta - \kappa_{\beta\gamma}n^\alpha),$$

and supposing that the discontinuity in the torsion terms obey the same rule as the discontinuity of $[g_{\alpha\beta,\gamma}]$, i. e., that $[T^\alpha_{\beta\gamma}] = \zeta^\alpha_\beta n_\gamma$, it reads

$$[K^\alpha_{\beta\gamma}] = \frac{1}{2}(\zeta^\alpha_\beta n_\gamma + \zeta^\alpha_\gamma n_\beta - \zeta^\alpha_\beta n_\gamma). \quad (4)$$

The components $\kappa_{\rho\sigma}$ emulate an intrinsic property of the brane itself. The torsion is continuous along the brane, and if there is some discontinuity, it is proportional to the extra dimension. Such proportionality is given, in principle, by another quantity ζ^α_β related to the brane. After these considerations, it follows that

$$[\Gamma^\alpha_{\beta\gamma}] = \frac{1}{2}((\kappa^\alpha_\beta + \zeta^\alpha_\beta - \zeta^\alpha_\beta)n_\gamma + (\kappa^\alpha_\gamma + \zeta^\alpha_\gamma)n_\beta - \kappa_{\beta\gamma}n^\alpha),$$

and hence

$$\begin{aligned} A^\alpha_{\beta\gamma\delta} &= \frac{1}{2}(\kappa^\alpha_\delta n_\beta n_\gamma - \kappa^\alpha_\gamma n_\beta n_\delta - \kappa_{\beta\delta}n^\alpha n_\gamma + \kappa_{\beta\gamma}n^\alpha n_\delta) \\ &+ \frac{1}{2}(\zeta^\alpha_\delta n_\beta n_\gamma - \zeta^\alpha_\gamma n_\beta n_\delta). \end{aligned} \quad (5)$$

Denoting $\kappa = \kappa^\alpha_\alpha$ and $\zeta = \zeta^\beta_\beta$, and suitably contracting two indices, it reads

$$\begin{aligned} A_{\beta\delta} &= \frac{1}{2}(\kappa^\alpha_\delta n_\beta n_\alpha - \kappa n_\beta n_\delta - \kappa_{\beta\delta} + \kappa_{\beta\alpha}n^\alpha n_\delta) \\ &+ \frac{1}{2}(\zeta^\alpha_\delta n_\beta n_\alpha - \zeta n_\beta n_\delta), \end{aligned} \quad (6)$$

and also

$$A = g^{\beta\delta}A_{\beta\delta} = (\kappa_{\alpha\delta}n^\alpha n^\delta - \kappa) + \frac{1}{2}(\zeta_{\alpha\delta}n^\alpha n^\delta - \zeta).$$

The δ -function part of the Einstein tensor $G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is given by

$$\begin{aligned} S_{\beta\delta} &= A_{\beta\delta} - \frac{1}{2}g_{\beta\delta}A \\ &= \frac{1}{2}(\kappa^\alpha_\delta n_\beta n_\alpha - \kappa n_\beta n_\delta - \kappa_{\beta\delta} + \kappa_{\beta\alpha}n^\alpha n_\delta \\ &\quad - g_{\beta\delta}(\kappa_{\rho\sigma}n^\rho n^\sigma - \kappa)) + \frac{1}{2}(\zeta^\alpha_\delta n_\beta n_\alpha - \zeta n_\beta n_\delta) \\ &\quad - \frac{1}{4}g_{\beta\delta}(\zeta_{\rho\sigma}n^\rho n^\sigma - \zeta). \end{aligned} \quad (7)$$

On the other hand, the total stress-energy tensor has the form

$$\pi_{\alpha\beta}^{\text{total}} = \Theta(\ell)\pi_{\alpha\beta}^+ + \Theta(-\ell)\pi_{\alpha\beta}^- + \delta(\ell)\pi_{\alpha\beta},$$

where $\pi_{\alpha\beta}^+$ and $\pi_{\alpha\beta}^-$ represent the bulk stress-energy in the regions where $\ell > 0$ and $\ell < 0$ respectively, while $\pi_{\alpha\beta}$ denotes the stress-energy localized on the hypersurface

³ Basically, if the condition $[g_{\alpha\beta}] = 0$ is not imposed, there appears the product $\Theta\delta$, which is not well defined in the Levi-Civita part of the connection.

Σ itself. From the Einstein equations, it follows that $\pi_{\alpha\beta} = (G_N)^{-1} S_{\alpha\beta}$.

Note that, since $\pi_{\alpha\beta}$ is tangent to the brane, it follows that $\pi_{\alpha\beta} n^\beta = 0$. However, from Eq.(7) the following equation

$$\begin{aligned} 4G_N \pi_{\alpha\beta} n^\beta &= \frac{1}{2}(\zeta_{\rho\sigma} n^\rho n^\sigma - \zeta) n_\alpha \\ &= -\frac{1}{2} \zeta_{\rho\sigma} q^{\rho\sigma} n_\alpha, \end{aligned} \quad (8)$$

is derived, which means that, in order to keep the consistency of the formalism, one has to impose $\zeta_{\rho\sigma} q^{\rho\sigma} = 0$, and the last term of Eq.(7) vanishes. Note that $\pi_{\alpha\beta}$ can be expressed by $\pi_{ab} = \pi_{\alpha\beta} h_a^\alpha h_b^\beta$, just using the h_a^α vierbein introduced in the previous Section. So, taking into account that $\pi_{\alpha\beta} = (G_N)^{-1} S_{\alpha\beta}$ and Eq.(7), it reads [14]

$$4G_N \pi_{ab} = -\kappa_{\alpha\beta} h_a^\alpha h_b^\beta + q^{rs} \kappa_{\mu\nu} h_r^\mu h_s^\nu q_{ab}. \quad (9)$$

Finally, relating the jump in the extrinsic curvature to $\kappa_{\rho\sigma}$, via the covariant derivative associated to $q_{\alpha\beta}$, the following expression can be obtained from Eq.(4):

$$\begin{aligned} [\nabla_\alpha n_\beta] &= \frac{1}{2}(\kappa_{\alpha\beta} - \kappa_{\gamma\alpha} n_\beta n^\gamma - \kappa_{\gamma\beta} n_\alpha n^\gamma) \\ &+ \frac{1}{2}(\zeta_\alpha^\gamma n_\beta + \zeta_\beta^\gamma n_\alpha - \zeta_\alpha^\gamma n_\beta) n_\gamma. \end{aligned} \quad (10)$$

However, it is clear that this jump of the extrinsic curvature across the brane, $[\nabla_\alpha n_\beta] \equiv [\Xi_{\alpha\beta}]$, can be also decomposed in terms of h_a^α vectors, leading to

$$[\Xi_{ab}] = \frac{1}{2} \kappa_{\alpha\beta} h_a^\alpha h_b^\beta. \quad (11)$$

Hence, after all, it follows that

$$2G_N \pi_{ab} = -[\Xi_{ab}] + [\Xi] q_{ab}. \quad (12)$$

It means that the second matching condition is absolutely the same that is valid without any torsion term. So, there is no difference in both junctions conditions within the context of a Riemann-Cartan manifold, which is an unexpected characteristic, since the torsion terms are directly related to the extrinsic curvature ($\nabla_\alpha n_\beta$) and effectively modify the connection.

IV. THE PROJECTED EQUATIONS ON THE BRANE

We have investigated the matching conditions in the presence of torsion terms, and under the assumptions of discontinuity across the brane. Surprisingly, both junctions conditions are shown to be the same as the usual case ($\Gamma_{\alpha\beta}^\rho = \overset{\circ}{\Gamma}_{\alpha\beta}^\rho$). We remark that, since the covariant derivative changes by torsion, the extrinsic curvature is also modified, and then the conventional arguments point in the direction of some modification in the matching conditions. However, it seems that the rôle of torsion

terms in the braneworld picture is restricted to the geometric part of effective Einstein equation on the brane. More explicitly, looking at the equation that relates the Einstein equation in four dimensions with bulk quantities (see, for example [8]) we have

$$\begin{aligned} {}^{(4)}G_{\rho\sigma} &= \frac{2k_5^2}{3} \left(T_{\alpha\beta} q_\rho^\alpha q_\sigma^\beta + (T_{\alpha\beta} n^\alpha n^\beta - \frac{1}{4} T) q_{\rho\sigma} \right) \\ &+ \Xi \Xi_{\rho\sigma} - \Xi_\rho^\alpha \Xi_{\alpha\sigma} - \frac{1}{2} q_{\rho\sigma} (\Xi^2 - \Xi^{\alpha\beta} \Xi_{\alpha\beta}) \\ &- {}^{(5)}C_{\beta\gamma\epsilon}^\alpha n_\alpha n^\gamma q_\rho^\beta q_\sigma^\epsilon, \end{aligned} \quad (13)$$

where $T_{\rho\sigma}$ denotes the energy-momentum tensor, $\Xi_{\rho\sigma} = q_\rho^\alpha q_\sigma^\beta \nabla_\alpha n_\beta$ is the extrinsic curvature, k_5 denotes the 5-dimensional gravitational constant, and ${}^{(5)}C_{\beta\rho\sigma}^\alpha$ denotes the Weyl tensor. By restricting to quantities evaluated on the brane, or tending to the brane, we see that the only way to get some contribution from torsion terms is via the term ${}^{(4)}G_{\rho\sigma}$, and also via the Weyl tensor. In the light of Section III it does not intervene in the extrinsic curvature tending to the brane. Actually, this fact makes the calculations easier when one tries to apply it to a particular model, specially possessing \mathbb{Z}_2 -symmetry, to extract more information about the rôle of the torsion in gravitational systems considered in braneworld scenarios.

In order to explicit the influence of (con)torsion terms in the projected equations on the brane, we shall to complete the GC program, from five to four dimensions, to the case with torsion. Note the by imposing the \mathbb{Z}_2 -symmetry, the extrinsic curvature reads

$$\Xi_{\alpha\beta}^+ = -\Xi_{\alpha\beta}^- = -2G_N \left(\pi_{\alpha\beta} - \frac{q_{\alpha\beta} \pi_\gamma^\gamma}{4} \right), \quad (14)$$

in such way that Eq.(12) reads⁴

$$\Xi_{\alpha\beta} = -G_N \left(\pi_{\alpha\beta} - \frac{q_{\alpha\beta} \pi_\gamma^\gamma}{4} \right). \quad (15)$$

Decomposing the stress-tensor associated with the bulk⁵ in $T_{\alpha\beta} = -\Lambda g_{\alpha\beta} + \delta S_{\alpha\beta}$ and $S_{\alpha\beta} = -\lambda q_{\alpha\beta} + \pi_{\alpha\beta}$, where Λ is the bulk cosmological constant and λ the brane tension, and substituting into Eq.(13) it follows after some algebra⁶,

$${}^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_N \pi_{\mu\nu} + k_5^4 Y_{\mu\nu} - E_{\mu\nu}, \quad (16)$$

where $E_{\mu\nu} = {}^{(5)}C_{\beta\gamma\sigma}^\alpha n_\alpha n^\gamma q_\mu^\beta q_\nu^\sigma$ encodes the Weyl tensor contribution, $G_N = \frac{\lambda k_5^4}{48\pi}$ is the analogous of the Newton gravitational constant, the tensor $Y_{\mu\nu}$ is quadratic in

⁴ Hereon, we remove the + and - labels.

⁵ Note that the delta factor appearing in $T_{\alpha\beta} = -\Lambda g_{\alpha\beta} + \delta S_{\alpha\beta}$ is necessary here, in order to localize the brane. In fact, this type of decomposition is compatible with the Israel-Darmois junction conditions. We remark that such a delta term can lead to problems in a more complete cosmological scenario, but for the purpose of this work there is not problem.

⁶ See, please, reference [8] for all the details.

the brane stress-tensor and given by $Y_{\mu\nu} = -\frac{1}{4}\pi_{\mu\alpha}\pi_{\nu}^{\alpha} + \frac{1}{12}\pi_{\gamma}^{\gamma}\pi_{\mu\nu} + \frac{1}{8}q_{\mu\nu}\pi_{\alpha\beta}\pi^{\alpha\beta} - \frac{1}{2}q_{\mu\nu}(\pi_{\gamma}^{\gamma})^2$ and $\Lambda_4 = \frac{k_5^2}{2}(\Lambda + \frac{1}{6}k_5^2\lambda^2)$ is the effective brane cosmological constant.

Now, the contributions arising from the (con)torsion terms are explicated in details. It is well known that the Riemann and Ricci tensors, and the curvature scalar written in terms of torsion are related with their partners, constructed with the usual metric compatible Levi-Civita connection by [20]

$$R_{\tau\alpha\beta}^{\lambda} = \mathring{R}_{\tau\alpha\beta}^{\lambda} + \nabla_{\alpha}K_{\tau\beta}^{\lambda} - \nabla_{\beta}K_{\tau\alpha}^{\lambda} + K_{\gamma\alpha}^{\lambda}K_{\tau\beta}^{\gamma} - K_{\gamma\beta}^{\lambda}K_{\tau\alpha}^{\gamma}, \quad (17)$$

$$R_{\tau\beta} = \mathring{R}_{\tau\beta} + \nabla_{\lambda}K_{\tau\beta}^{\lambda} - \nabla_{\beta}K_{\tau\lambda}^{\lambda} + K_{\gamma\lambda}^{\lambda}K_{\tau\beta}^{\gamma} - K_{\tau\gamma}^{\lambda}K_{\lambda\beta}^{\gamma} \quad (18)$$

and

$$R = \mathring{R} + 2\nabla^{\lambda}K_{\lambda\tau}^{\tau} - K_{\tau\lambda}^{\lambda}K^{\tau\lambda}_{\lambda} + K_{\tau\gamma\lambda}K^{\tau\lambda\gamma}, \quad (19)$$

where the quantities \mathring{X} are constructed with the usual metric compatible Levi-Civita connection, and ∇ denotes the covariant derivative *without* torsion. Clearly such relations holds in any dimension. Therefore, by denoting D_{μ} the covariant 4-dimensional derivative acting on the brane, it is easy to see that, from Eqs.(17),(18), and (19),

the Einstein tensor on the brane is given by

$$\begin{aligned} {}^{(4)}G_{\mu\nu} = & {}^{(4)}\mathring{G}_{\mu\nu} + D_{\lambda}{}^{(4)}K_{\mu\nu}^{\lambda} - D_{\nu}{}^{(4)}K_{\mu\lambda}^{\lambda} \\ & + {}^{(4)}K_{\gamma\lambda}^{\lambda}{}^{(4)}K_{\mu\nu}^{\gamma} - {}^{(4)}K_{\mu\gamma}^{\lambda}{}^{(4)}K_{\lambda\nu}^{\gamma} \\ & - q_{\mu\nu}\left(D^{\lambda}{}^{(4)}K_{\lambda\tau}^{\tau} + \frac{1}{2}{}^{(4)}K_{\tau\lambda}^{\lambda}{}^{(4)}K^{\tau\gamma}_{\gamma}\right. \\ & \left. + \frac{1}{2}{}^{(4)}K_{\tau\gamma\lambda}{}^{(4)}K^{\tau\lambda\gamma}\right). \end{aligned} \quad (20)$$

Note the appearance of terms multiplying the brane metric. As it shall be seen, these terms compose a new effective cosmological constant.

The $E_{\mu\nu}$ tensor can be expressed in terms of the bulk contorsion terms ($K^{\mu}_{\nu\alpha}$) by

$$\begin{aligned} E_{\kappa\delta} = & \mathring{E}_{\kappa\delta} + \left(\nabla_{\nu}K^{\mu}_{\alpha\beta} - \nabla_{\beta}K^{\mu}_{\alpha\nu} + K^{\mu}_{\gamma\nu}K^{\gamma}_{\alpha\beta} \right. \\ & \left. - K^{\mu}_{\gamma\beta}K^{\gamma}_{\alpha\nu}\right)n_{\mu}n^{\nu}q_{\kappa}^{\alpha}q_{\delta}^{\beta} - \frac{2}{3}(q_{\kappa}^{\alpha}q_{\delta}^{\beta} + n^{\alpha}n^{\beta}q_{\kappa\delta}) \\ & \times \left(\nabla_{\lambda}K^{\lambda}_{\beta\alpha} - \nabla_{\alpha}K^{\lambda}_{\beta\lambda} + K^{\lambda}_{\gamma\lambda}K^{\gamma}_{\beta\alpha} - K^{\sigma}_{\beta\gamma}K^{\gamma}_{\sigma\alpha}\right) \\ & + \frac{1}{6}q_{\kappa\delta}\left(2\nabla^{\lambda}K_{\lambda\tau}^{\tau} - K_{\tau\lambda}^{\lambda}K^{\tau\gamma}_{\gamma} + K_{\tau\gamma\lambda}K^{\tau\lambda\gamma}\right) \end{aligned} \quad (21)$$

where ∇_{μ} is the bulk covariant derivative. Now, the explicit influence of the contorsion terms in the Einstein brane equation can be appreciated. From Eqs.(16), (20), and (21), it reads

$$\begin{aligned} & {}^{(4)}\mathring{G}_{\mu\nu} + D_{\lambda}{}^{(4)}K_{\mu\nu}^{\lambda} - D_{\nu}{}^{(4)}K_{\mu\lambda}^{\lambda} + {}^{(4)}K^{\delta}_{\gamma\delta}{}^{(4)}K_{\mu\gamma}^{\lambda}{}^{(4)}K^{\gamma}_{\lambda\nu} = -\tilde{\Lambda}_4 q_{\mu\nu} + 8\pi G_N \pi_{\mu\nu} + k_5^4 Y_{\mu\nu} - \mathring{E}_{\mu\nu} + q_{\mu}^{\alpha}q_{\nu}^{\beta} \\ & \times \left[\frac{2}{3}\left(\nabla_{\lambda}K^{\lambda}_{\beta\alpha} - \nabla_{\alpha}K^{\lambda}_{\beta\lambda} + K^{\sigma}_{\gamma\sigma}K^{\gamma}_{\beta\alpha} - K^{\lambda}_{\beta\gamma}K^{\gamma}_{\lambda\alpha}\right) - n_{\rho}n^{\sigma}\left(\nabla_{\sigma}K^{\rho}_{\alpha\beta} - \nabla_{\beta}K^{\rho}_{\alpha\sigma} + K^{\rho}_{\gamma\sigma}K^{\gamma}_{\alpha\beta} - K^{\rho}_{\gamma\beta}K^{\gamma}_{\alpha\sigma}\right)\right] \end{aligned} \quad (22)$$

where the new effective cosmological constant is given by

$$\begin{aligned} \tilde{\Lambda}_4 \equiv & \Lambda_4 - D^{\lambda}{}^{(4)}K_{\lambda\tau}^{\tau} + \frac{1}{2}{}^{(4)}K_{\tau\alpha}^{\alpha}{}^{(4)}K^{\tau\lambda}_{\lambda} - \frac{1}{2}{}^{(4)}K_{\tau\gamma\lambda}{}^{(4)}K^{\tau\lambda\gamma} - \frac{2}{3}n^{\alpha}n^{\beta}\left(\nabla_{\lambda}K^{\lambda}_{\beta\alpha} - \nabla_{\alpha}K^{\lambda}_{\beta\lambda} \right. \\ & \left. + K^{\lambda}_{\gamma\lambda}K^{\gamma}_{\beta\alpha} - K^{\sigma}_{\beta\gamma}K^{\gamma}_{\sigma\alpha}\right) + \frac{1}{6}\left(2\nabla^{\lambda}K_{\lambda\tau}^{\tau} - K_{\tau\alpha}^{\alpha}K^{\tau\lambda}_{\lambda} + K_{\tau\gamma\lambda}K^{\tau\lambda\gamma}\right). \end{aligned} \quad (23)$$

Eqs. (22) and (23) enclose the main result of this paper. From Eq.(22) it follows that the factors involving both contorsion, in four and in five dimensions, change drastically the effective Einstein equation on the brane, as well as the effective cosmological constant. We shall comment this important and remarkable result in the next Section.

V. CONCLUDING REMARKS AND OUTLOOK

There are some alternative derivations of the junction conditions for a brane in a 5-dimensional bulk, when

Gauss-Bonnet equations are used to describe gravity [21]. Also, Israel junction conditions can be generalized for a wider class of theories by direct integration of the field equations, where a specific non-minimal coupling of matter to gravity suggests promising classes of braneworld scenarios [22]. In addition, it is also possible to generalize matching conditions for cosmological perturbations in a teleparallel Friedmann universe, following the same lines as [23].

In the case studied here, however, the matching conditions are not modified by the inclusion of torsion terms in the connection. As noted, it is a remarkable and unex-

pected characteristic. Besides, all the development concerning the formalism presented is accomplished in the context of braneworld models. In such framework, the appearance of torsion terms is quite justifiable. However, the fact that the matching conditions remain unalterable in the presence of torsion is still valid in usual 4-dimensional theories.

Once investigated the junction conditions, we have obtained, via Gauss-Codazzi formalism, the Einstein effective projected equation on the brane. If, on one hand, the torsion terms do not intervenes in the usual Israel-Darmois conditions, on the other hand it modifies drastically the brane Einstein equations. Eq.(23) shows up the strong dependence of the new effective cosmological constant on the four and five-dimensional contorsion terms. It reveals promising possibilities. For instance, by a suitable behavior of such new terms, $\tilde{\Lambda}_4$ can be very small. In a more complete scenario, $\tilde{\Lambda}_4$ could be not even a constant. It must be stressed that these types of modification in the projected Einstein equation also appear in other models in modified gravity [24].

This paper intends to give the necessary step in order to formalize the mathematical implementation of torsion terms in braneworld scenarios. The application of our results are beyond the scope of this work. We finalize, however, pointing out some interesting research lines coming from the use of the results — obtained in this paper — in cosmological problems.

The final result is very important from the cosmolog-

ical viewpoint. It is clear that deviations of the usual braneworld cosmology can be obtained from the analysis of phenomenological systems in the light of Eq.(22). Physical aspects, more specifically the analysis of cosmological signatures as found in ref. [25], arising from the combination of the extra dimensions and torsion should be systematically investigated and compared with usual braneworld models. The ubiquitous presence of torsion terms leads, by all means, to subtle but important deviations of usual braneworlds in General Relativity. For instance, the equation (22) can be used as a starting point to describe the flat behavior of galaxy rotational curves without claim for dark matter. This last problem was already analyzed in the context of brane worlds [26], however the outcome arising from the torsion terms has never been investigated. A systematic comparative study between usual braneworld models and those braneworld models embedded in an Einstein-Cartan manifold is, potentially, interesting since it can lead us to new branches inside brane physics. We shall address to those questions in the future.

Acknowledgment

The authors thank to Prof. R. A. Mosna and the CQG Referee for fruitful suggestions. J. M. Hoff da Silva thanks to CAPES-Brazil for financial support.

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